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A note on reduction of quadratic and bilinear programs with equality constraints

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Abstract. Reduction of some classes of global optimization programs to bilinear programs may be done in various ways, and the choice of method clearly influences the ease of solution of the resulting problem. In this note we show how linear equality constraints may be used together with graph theoretic tools to reduce a bilinear program, i.e., eliminate variables from quadratic terms to minimize the number of complicating variables. The method is illustrated on an example. Computer results are reported on known test problems.

Key words: Quadratic program, Bilinear program, Reduction, Gaussian pivoting

1. Introduction

A **quadratic program**, in its most general form, may be written as follows:

$$\text{Problem } (Q) \begin{cases} \min \displaystyle\sum_{i=1}^{n} \sum_{j=i}^{n} q_{ij} x_i x_j + \sum_{i=1}^{n} q_i x_i + q_0 \\ \text{subject to:} \\ \displaystyle\sum_{i=1}^{n} \sum_{j=i}^{n} r_{ij}^k x_i x_j + \sum_{i=1}^{n} r_i^k x_i + r_0^k \leqslant 0 \quad k = 1, 2, \ldots, \ell \\ x_j \in \mathbb{R} \quad j = 1, 2, \ldots, n \end{cases}$$

where the coefficients $q_{ij},\ q_i,\ q_0,\ r_{ij}^k,\ r_i^k,\ r_0^k \quad (i, j = 1, 2, \ldots, n;\ j \geqslant i;\ k = 1, 2, \ldots, \ell)$ are real numbers. No assumptions are made on convexity or concavity of the objective function or of the constraints. The constraints possibly include nonnegativity and/or range constraints. Without loss of generality, some of the constraints of (Q) may be assumed to be equalities and/or linear.

Floudas et al. [10] reduced (Q) to a general **bilinear program** (B) by duplication of variables. Bilinear programs may be written as follows:

$$\text{Problem } (B) \begin{cases} \min \quad \sum_{i=1}^{n} \sum_{j=1}^{p} c_{ij} x_i y_j + \sum_{i=1}^{n} c'_i x_i + \sum_{i=1}^{p} c''_i y_i + c_0 \\ \text{subject to:} \\ \sum_{i=1}^{n} \sum_{j=1}^{p} a^k_{ij} x_i y_j + \sum_{i=1}^{n} a'^k_i x_i + \sum_{i=1}^{p} a''^k_i y_i + a^k_o \leqslant 0 \quad k = 1, 2, \ldots, \ell \\ x_i \in \mathbb{R} \quad i = 1, 2, \ldots, n \\ y_i \in \mathbb{R} \quad i = 1, 2, \ldots, p \end{cases}$$

where the coefficients $c_{ij},\ c'_i,\ c''_j,\ c_0,\ a^k_{ij},\ a'^k_i,\ a''^k_j,\ a^k_0 \quad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, p;\ k = 1, 2, \ldots, \ell)$ are real numbers. Again the constraints may include nonnegativity and/or range ones, as well as (linear) equalities. When any one set of variables is fixed (the x_i or the y_i), a linear program is obtained.

Many algorithms have been suggested for solving problems (Q) and (B). Nonconvex quadratic programming is surveyed in the landmark book of Horst and Tuy [20] on *Global Optimization*, in the *Introduction to Global Optimization* of Horst et al. [18], in the new book of Floudas *Deterministic Global Optimization* [8] and in the chapter on *Quadratic Optimization* by Floudas and Visweswaran [14] in the *Handbook of Global Optimization* edited by Horst and Pardalos [17]. Recent algorithms for problem (Q) include simplicial branch-and-bound ones due to Horst and Thoai [19] and Raber [24], duality bound methods of Ben-Tal et al. [6] and Thoai [29], a relaxation method of Al-Khayyal et al. [2], reformulation-linearization techniques of Sherali and Tuncbilek [27, 28] and a branch-and-cut algorithm of Audet et al. [5]. Kojima and Tuncel [21] present successive convex relaxation methods based on semidefinite and semi-infinite linear programming. Further references may be found in these books and papers.

Bilinear programming is discussed in the books and chapters cited above, as well as in a survey of Al-Khayyal [1] and the book of Konno et al. [22] on *Optimization on low rank nonconvex structures* and the recent paper of Audet et al. [4]. Problem (Q) can be solved with the algorithm proposed for problem (B) or, more efficiently by algorithms exploiting its particular structure. There include a primal-relaxed dual approach of Floudas and Visweswaran [12, 13] close to generalized Benders decomposition [15, 25, 30]), a relaxation-linearization method of Sherali and Alameddine [26] and a projection and branch-and-bound algorithm of Quesada and Grossmann [23].

It is well known that mathematical programs can often be written in different forms, which yield the same optimal solution, but which may vary considerably in the difficulty of their resolution. A good example is a recent study by Audet et al. [3] of the *pooling problem*, a bilinear program arising in the oil industry. Two formulations, based on flows and proportions are presented there. For large instances, resolution time with the algorithm of [5] is of over 1 h for one formulation and a few tens of seconds for the other.

So when reformulating a quadratic program as a bilinear one, one should aim at an easy to solve formulation. Difficulty of resolution appears to increase with the number of *complicating variables*, i.e., variables in the smallest of the two sets and with the number of *bilinear terms*.

Let us recall some definitions. Let $G = (V, E)$ denote a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set E. A set $S \subseteq V$ of vertices is a **stable** (or independent) set if any two vertices in S are not adjacent. A set $T \subseteq V$ of vertices is a **transversal** (or vertex cover) if any edge of E contains at least one vertex of T. The complement in V of a stable set S of G is a transversal T of G. A graph G is **bipartite** if its vertex set V can be partitioned into two sets V_1 and V_2 such that any edge of E joins a vertex of V_1 to a vertex of V_2. The **co-occurrence graph** G of problem (Q) has vertices associated with the variables of the quadratic program and edges joining vertices associated with variables appearing jointly in one term of the objective function or constraints. The co-occurrence graph of problem (B) is bipartite.

Hansen and Jaumard [16] have proven that any transversal T_i of the co-occurrence graph G allows reformulation of (Q) to a bilinear program (B_i), with a number $|T_i|$ of complicating variables. Then a minimum set of complicating variables corresponds to a minimum transversal of the co-occurrence graph $(\min_i\{|T_i|\})$.

In this note we show how linear equality constraints can be used to eliminate variables and reduce more the bilinear program. Making such constraints explicit (Q) can be written:

$$\text{Problem } (Q') \begin{cases} \min \displaystyle\sum_{i=1}^{n} \sum_{j=i}^{n} q_{ij} x_i x_j + \sum_{i=1}^{n} q_i x_i + q_0 \\ \text{subject to:} \\ \displaystyle\sum_{i=1}^{n} \sum_{j=i}^{n} r_{ij}^k x_i x_j + \sum_{i=1}^{n} r_i^k x_i + r_0^k \leqslant 0 \quad k = 1, 2, \dots, \ell \\ \displaystyle\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \dots, m \\ x_j \in \mathbb{R} \quad j = 1, 2, \dots, n \end{cases}$$

where a_{ij} and b_i $(i = 1, \dots, m,\ j = 1, \dots, n)$ are real numbers. When reformulating (Q') as a bilinear program, the minimum number of complicating variables is found as $\min_i\{|T_i| - \mu_i\}$ (instead of $\min_i\{|T_i|\}$), where μ_i is the maximum number of linearly independent equations *with all variables from* T_i. An algorithm that uses *Gaussian partial pivoting* to provide eliminations and substitutions is described. Computer results are reported on known test examples.

2. Exploiting linear equality constraints

Let us denote by $G = (V, E)$ the co-occurrence graph of quadratic program (Q) or (Q'), with T_i being any (minimal) transversal of G, and with $\beta_i(G) = |T_i|$, the cardinality of set T_i, $i = 1, \ldots, r$, where r is the total number of transversals of G. Since T_i is a (minimal) transversal set, quadratic program (Q) has a reduction to a bilinear program (B_i) with $\beta_i = \beta_i(G)$ complicating variables (*Theorem 2* in [16]). Let us denote by $G_i = (V_i, E_i)$, $i = 1, \ldots, r$, the co-occurrence graphs of the associated bilinear programs. The following holds:

PROPOSITION 1. *The number of complicating variables β_i of bilinear program (B_i) with co-occurrence graph $G_i = (V_i, E_i)$ can be reduced by one if there is one linear equation with not all zero coefficients, whose vertices (associated with variables) all belong to T_i.*

Proof. Since G_i is obtained by a reduction of (Q), its vertex set is partitioned into two sets T_i and S_i such that any edge of E_i joins a vertex of T_i to a vertex of S_i, i.e., G_i is *bipartite*. Without loss of generality, let us suppose that $T_i = \{v_1, \ldots, v_\beta\}$, and $(a_1 \neq 0)$ $x_1 = b - a_2x_2 - \cdots - a_\beta x_\beta$. Let $T_i' = T_i \setminus \{v_1\}$, $V_i' = T_i' \cup S_i$, and $G_i' = (V_i', E_i')$. It is sufficient to prove that the set $T_i' = T_i \setminus \{v_1\}$ is a transversal of co-occurrence graph G_i', obtained after elimination of variable x_1 from bilinear program (B_i), i.e., after deleting vertex v_1 from T_i and after substituting x_1 in all bilinear terms where it appears. Then the result, $|T_i'| = |T_i| - 1 = \beta_i - 1$, holds. In fact, since G_i is bipartite, substituting x_1 into any bilinear term where it appears gives:

$$y_\alpha x_1 = by_\alpha - \sum_{j=2}^{\beta} a_j x_j y_\alpha, \;\; y_\alpha \in S_i.$$

It is now obvious that all possible new edges of G_i' (associated with $x_j y_\alpha$) will join a vertex of T_i' to a vertex of S_i. In other words, there is no new edge with both extremities belonging to S_i. Thus, G_i' is bipartite and T_i' is (one of) its (minimal) transversal(s). □

The extension of this result is given in the following proposition.

PROPOSITION 2. *The number of complicating variables β_i of bilinear program (B_i) with co-occurrence graph $G_i = ((T_i, S_i), E_i)$, can be reduced by μ_i, if there are μ_i linearly independent equations, whose vertices (associated with variables) all belong to T_i.*

Proof. $\mu_i = 1$ is proved in *Proposition 1*. The result then follows by induction. □

From the previous Propositions it appears that using linear equations, $Ax = b$, to find the minimum number of complicating variables of the given bilinear programs B_i with bipartition T_i and S_i ($|T_i| \leqslant |S_i|$), $i = 1, \ldots, r$, is tantamount to finding the maximum number of linearly independent equations with all variables from the smallest of the two sets (T_i). That fact is given in the following Theorem.

THEOREM 1. *A quadratic program (Q') with co-occurrence graph G, its r transversals T_i and $\beta_i = |T_i| (i = 1, \ldots, r)$, has a reduction to a bilinear program (B') with* $\min_i \{\beta_i - \mu_i\}$ *complicating variables, where μ_i is the maximum number of linearly independent equations with all indices from T_i.*

We shall now explain how to compute μ_i for every transversal $T_i, i = 1, \ldots, r$. Let us interchange the columns of the linear system $Ax = b$, to get

$$[A_1^i \ A_2^i] \cdot \begin{bmatrix} x^i \\ y^i \end{bmatrix} = b,$$

where A_1^i and A_2^i are sub-matrices of A whose columns belong to index sets S_i and T_i, respectively. If the submatrix A_1^i has full row rank m, then it is not possible to derive equations with variables from T_i only, i.e., the linear equality constraints cannot be used in reducing the number of complicating variables. Let us assume that $\rho_i = rank(A_1^i) < m$. Then, by using any linear system solution method (for instance the *Gauss method*), we get the partition

$$\begin{bmatrix} A_{11}^i & A_{12}^i \\ 0 & A_{22}^i \end{bmatrix} \cdot \begin{bmatrix} x^i \\ y^i \end{bmatrix} = \begin{bmatrix} b_1^i \\ b_2^i \end{bmatrix},$$

where A_{11}^i is an upper triangular $\rho_i \times (n - \beta_i)$ matrix, A_{12}^i is $\rho_i \times \beta_i$, and A_{22}^i is $(m - \rho_i) \times \beta_i$. Thus, we have $\mu_i = rank(A_{22}^i)$.

Let us assume that i^* is found such that $\beta_{i^*} - \mu_{i^*} = \min_i\{\beta_i - \mu_i\}$, and let $\beta = \beta_{i^*}, \ \mu = \mu_{i^*}$. We have the system $Ax = b$ (obtained after transformations) corresponding to bipartition T_{i^*} and S_{i^*} as

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

With this best among r minimal transversals, we now have to eliminate μ variables from the system $A_{22}y = b_2$. The last system can be again transformed into $A'_{22}y' + A''_{22}y'' = b'_2$, i.e.,

$$\begin{bmatrix} A_{11} & A'_{12} & A''_{12} \\ 0 & A'_{22} & A''_{22} \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y' \\ y'' \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ 0 \end{bmatrix}.$$

where A'_{22} is an upper triangular $\mu \times \mu$ matrix. The solution of the last triangular system

$$y' = A' - 1_{22}(b'_2 - A''_{22}y'') \tag{1}$$

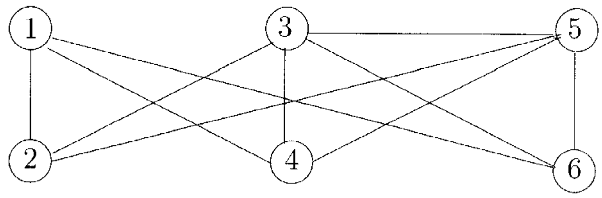

Figure 1. Co-occurrence graph, G.

can be substituted in

$$A_{11}x + A'_{12}y' + A''_{12}y'' = b_1. \tag{2}$$

Thus we obtain the new set of linear constraints as

$$A_{11}x + (A''_{12} - A'_{12}A'^{-1}_{22}A''_{22})y'' = b_1 - A'_{12}A'^{-1}_{22}b'_2. \tag{3}$$

The programming aspects are described in detail in the Appendix to the first version of this note [7]. The following example illustrates the process.
EXAMPLE. Consider the following illustrative problem:

$$\min f(x) = (x_1 + x_3 + x_5)(x_2 + x_4 + x_6) + x_3x_5$$

subject to

$$\begin{array}{rcrcrcrcrcrcrcr} 3x_1 & + & 2x_2 & + & 4x_3 & + & 2x_4 & + & x_5 & + & 4x_6 & = & 9 \\ 4x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & + & x_5 & + & 2x_6 & = & 9 \\ 2x_1 & + & 4x_2 & + & 7x_3 & + & 4x_4 & + & 3x_5 & + & 8x_6 & = & 18. \end{array}$$

The co-occurrence graph is shown in Figure 1. Note that all six variables appear in nonlinear terms, of which there are a total of 10. The analysis for two transversals, $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_5, v_6\}$ is now summarized using the notation and procedure identified above.
(1) $T_1 = \{v_1, v_3, v_5\}$:

$$A_1^1 = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 3 & 2 \\ 4 & 4 & 8 \end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 3 & 4 & 1 \\ 4 & 1 & 1 \\ 2 & 7 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 9 \\ 18 \end{bmatrix}.$$

After transformations, we obtain

$$\left[A_1^1\ A_2^1\right] = \left[\begin{array}{rrr|rrr} 1 & 1 & 2 & 1.5 & 2 & 0.5 \\ 0 & 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right],$$

so that the rank of A_1^1, $\rho_1 = 2$, the rank of A_{22}^1, $\mu_1 = 1$, and the number of complicating variables becomes

$$z_1 = \beta_1 - \mu_1 = 3 - 1 = 2.$$

(2) $T_2 = \{v_2, v_4, v_5, v_6\}$:

$$A_1^2 = \begin{bmatrix} 3 & 4 \\ 4 & 1 \\ 2 & 7 \end{bmatrix}, \qquad A_2^2 = \begin{bmatrix} 2 & 2 & 1 & 4 \\ 2 & 3 & 1 & 2 \\ 4 & 4 & 3 & 8 \end{bmatrix};$$

readily, we obtain $\rho_2 = 2$, $\mu_2 = 1$, so that the number of complicating variables becomes

$$z_2 = \beta_2 - \mu_2 = 4 - 1 = 3.$$

Enumeration of the transversals of G shows that $\min_i\{\beta_i - \mu_i\} = 2$ ($i^* = 1$); that is, the minimum number of complicating variables is 2. Letting $y_1 = x_1$, $y_2 = x_3$, $y_3 = x_5$, solving $y_1 = -0.25y_2 + 0.25y_3$, and substituting leads to the following equivalent reduced bilinear program:

$$\min(0.75y_2 + 1.25y_3)(x_2 + x_4 + x_6) + y_2x_7$$

subject to

$$\begin{array}{rrrrrl} x_2 + x_4 & +2x_6 & & +1.625y_2 & +0.875y_3 & = 4.5 \\ x_4 & -2x_6 & & -3.25y_2 & +0.25y_3 & = 0 \\ & & x_7 & & -y_3 & = 0. \end{array}$$

Note that although the number of complicating variables cannot be reduced further, it may be possible to further simplify the problem. Consider the final form of the linear equality constraints in (3). If the coefficients of y'' are all zeros in any row, we can use that equation to eliminate a noncomplicating variable. This may in turn reduce the number of bilinear terms.

3. Computational results and conclusions

The reduction procedure was tested on several well-known problems from Floudas and Pardalos [11]. The results are summarized in Table 1. Column 1 gives the problem identification in [11]; the next two columns give the number of rows, m, and number of columns (variables), n, in the set of linear equality constraints (in all cases, n also equals the number of nodes in the co-occurrence graph); this is followed, respectively, by the number of edges ($|E|$) in the co-occurrence graph, the total number of minimal transversals (r), the number of complicating variables before reduction (β) and the number eliminated (μ), and finally the CPU time (s) to find the 'best' reduction. We see from the table that a substantial reduction is obtained in most of the problems investigated. It also appears that the number of minimal transversals is highly variable from problem to problem and may be quite large. In such a case, it is preferable (or imposed by time limit) to enumerate only some of them.

Table 1. Computational Results on Sun Sparc 10 Station.

Pr. #	m	n	$\|E\|$	r	β	μ	CPU time
12.2	5	12	6	48	6	3	0.05
5.2	13	48	44	400	18	6	9.89
5.4	17	38	36	64	14	10	1.63
5.6	22	110	96	512000	31	6	1067.12
7.2	17	42	26	1296	12	1	29.08
9.2	22	76	95	480	18	5	36.43
9.3	17	71	90	480	18	5	33.39
9.6	12	48	52	3744	16	0	50.07
10.2	41	102	78	103680	22	1	23247.80

To conclude, this note investigates the use of linear equality constraints to reduce general quadratic programs by eliminating complicating variables in the equivalent bilinear form. Significant reduction is obtained on several test problems from the literature. Future work will examine the properties of nonlinear equality constraints in the reduction process.

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